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Entanglement for the System of Two 2-Level Atoms Interacting with a Single-Mode Through Cooperative Interaction

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Abstract The interaction of two 2-level atoms inside a resonant microcavity under stimulated emission via multi-photon-transition is considered. An analytical solution for this system when both atoms are initially in the exited state and the field in a coherent state is obtained. Entanglement dynamics between the two atoms taking into account the effect of the stimulated emission is studied by using various measures of entanglement. We compare the results for these various measures, and discuss the entanglement induced due to the stimulated emission.

Keywords Entanglement · Two atoms · Linear entropy · Stimulated emission

1 Introduction

Quantum entanglement is one of the most profound features of quantum mechanics and has been considered to be a valuable physical resource in the field of quantum information science, with its different attributes [1–6]. The correlation between pairwise-entangled quantum states of two spatially separated particles is one of the most interesting features of quantum mechanics. Besides their applications, a great deal of interest has been intensively focused on designing and realizing possible quantum entangling proposals that can be essential ingredients in quantum communication [7–11] and quantum computation [12]. It can be realized in different physical systems such as trapped ions [13], spins in nuclear magnetic resonance [14], superconducting Josephsen junctions [15]. Cooper pairs in solid states quantum-dots [16, 17], and cavity quantum-electrodynamics systems [18]. Practically, two atoms can be entangled through continuous driving by a coherent pump field [19, 20], the assistance of a thermal cavity field [21–24], or even the inducement from the atomic spontaneous emission [22, 25–27]. Generally, these schemes are effective since there are some

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kinds of interaction between atom and atom or atom and cavity field. While, there is another interaction that is usually not included: the stimulated emission [28] of atoms in an atomic ensemble. The stimulated emission of atom emerges when atom in higher energy level is driven by a polarized photon [29]. Especially, when atomic population inversion is realized, in a laser system, the stimulated emission plays a key role in photon absorb-emission process. The emitted photon is treated as a new driving field acting on the atoms [30] since the emitted photon has the identical character with that of the driving photon. Consequently entanglement between atoms can be enhanced [31].

In this paper, we consider a system of two atoms interacting under the stimulated emission via multi-photon-transition. An analytical solution for this system when both atoms are initially in the exited state and the field in a coherent state is obtained. We study entanglement dynamics between the two atoms, interacting through cooperative interaction, by using various measures of entanglement such as the von Neumann entropy and linear entropy. The paper is arranged as follows: Sect. 2 is devoted to the physical system and its dynamics. In Sect. 3, we employ analytical results obtained in Sect. 2 to discuss various measures of entanglement for this system. Finally in Sect. 4, we present our conclusion.

2 The Model and Its Solution

We consider a system of a two 2-level atoms located in a microcavity with a single mode quantized cavity field. In this system, either atom can transit from its exited state $|e\rangle$ to its ground state $|g\rangle$ under the driving of a resonant laser field and emit polarized photons. Also, either atom that is in its state $|g\rangle$ can absorb such polarized photons and jump to its state $|e\rangle$. That is, besides a driving field E, the emitted polarized photons from one atom can also act as a new driving field $\acute{E}$ with respect to the other atom and vice versa. As a result, the whole cavity field is the sum of two fields E and $\acute{E}$. The authors in [30] discuss a similar model where they assume the atoms are coupled to a classical external field. The Hamiltonian of the system in the interaction picture reads [31]:

$$\hat{H} = \lambda_{drv} \sum_{i=1}^{2} (\hat{a}^k \hat{\sigma}_+^i + \hat{a}^{\dagger k} \hat{\sigma}_-^i) + \frac{\lambda_{stm}}{2} \sum_{i,j=1, i\neq j}^{2} [\hat{\sigma}_z^i (\hat{a}^k \hat{\sigma}_+^j + \hat{a}^{\dagger k} \hat{\sigma}_-^j) + (\hat{a}^k \hat{\sigma}_+^j + \hat{a}^{\dagger k} \hat{\sigma}_-^j)\hat{\sigma}_z^i]. \tag{1}$$

Such Hamiltonian is equivalent to

$$\hat{H} = \left[\lambda_{drv} + \lambda_{stm} \sum_{i=1, i\neq j}^{2} \hat{\sigma}_z^i\right] \sum_{j=1}^{2} (\hat{a}^k \hat{\sigma}_+^j + \hat{a}^{\dagger k} \hat{\sigma}_-^j), \tag{2}$$

where $\hat{a}$, $\hat{a}^\dagger$ are annihilation and creation operators of driving field, $\hat{\sigma}_+^i = |e\rangle_i \langle g|$, $\hat{\sigma}_-^i = |g\rangle_i \langle e|$ and $\hat{\sigma}_z^i = |e\rangle_i \langle e| - |g\rangle_i \langle g|$ are the transition operators and the inversion operator of atom i respectively, λ_{drv} represent the coupling strength between separate atom and field E and λ_{ste} denotes the interaction strength between atom j and $\acute{E}$. Generally $\lambda_{ste} = \alpha_s \lambda_{drv}$ and α_s is determined by the stimulated emission coefficient, atomic density and λ_{drv} and we take $g = (1 + \alpha_s)\lambda_{drv}$. When the two atoms are spatially much close to each other, α_s will fall in the range between 0 and 1 [30]. The initial state of the total atom-atom field system can be written as:

$$|\psi(0)\rangle = |\psi(0)\rangle_F \otimes |\psi(0)\rangle_A = \sum_{n=0} C_n |n, e, e\rangle \tag{3}$$

which means that each atom start in its exited state and the field is assumed to be initially in a coherent state, where $C_n = e^{-|\alpha^2|/2} \frac{\alpha^n}{\sqrt{n!}}$, $\alpha = |\alpha| e^{i\phi}$ and $|\alpha|^2$ is the mean photon number of the coherent field and ϕ is the phase angle of the coherent field (here we take $\phi = 0$). The solution of the Schödinger equation in the interaction picture i.e., the wave function of the system at any time $t > 0$ is given by:

$$|\Psi(\tau)\rangle = |\Psi_1\rangle|e,e\rangle + |\Psi_2\rangle|e,g\rangle + |\Psi_3\rangle|g,e\rangle + |\Psi_4\rangle|g,g\rangle \tag{4}$$

with

$$\begin{aligned} |\Psi_1\rangle &= \sum_{n=0}^{\infty} \psi_1(n,\tau)|n\rangle, \qquad |\Psi_2\rangle = \sum_{n=0}^{\infty} \psi_2(n,\tau)|n+k\rangle, \\ |\Psi_3\rangle &= \sum_{n=0}^{\infty} \psi_3(n,\tau)|n+k\rangle, \qquad |\Psi_4\rangle = \sum_{n=0}^{\infty} \psi_4(n,\tau)|n+2k\rangle, \end{aligned} \tag{5}$$

where $\tau = gt$ is the scaled time. The explicit forms for the dynamical coefficients $\psi_j(n,\tau)$ ($j = 1, 2, 3$ and 4) are:

$$\begin{aligned} \psi_1(n,\tau) &= \frac{1}{\zeta^2\gamma^2+\eta^2}\left[\zeta^2\gamma^2 + \eta^2 \cos\left(\tau\sqrt{2(\zeta^2\gamma^2+\eta^2)}\right)\right], \\ \psi_2(n,\tau) &= \psi_3(n,\tau) = \frac{-i\eta}{\sqrt{2(\zeta^2\gamma^2+\eta^2)}}\left[\sin\left(\tau\sqrt{2(\zeta^2\gamma^2+\eta^2)}\right)\right], \\ \psi_4(n,\tau) &= \frac{\zeta\gamma\eta}{\zeta^2\gamma^2+\eta^2}\left[\cos\left(\tau\sqrt{2(\zeta^2\gamma^2+\eta^2)}\right) - 1\right], \end{aligned} \tag{6}$$

where $\zeta = \sqrt{\frac{(n+2k)!}{(n+k)!}}$, $\eta = \sqrt{\frac{(n+k)!}{(n)!}}$ and $\gamma = \frac{1-\alpha_s}{1+\alpha_s}$. The properties of the quantum entanglement between two entangled atoms and a single mode coherent field are discussed in the next section using the above results.

3 The Quantum Entanglement Between the Two 2-Level Atoms and the Field

In this section, we investigate various measures of the degree of entanglement between the field and the two atoms for the system under consideration.

3.1 von Neumann Entropy

We use the von Neumann entropy as a measure of the degree of entanglement between the field and the two atoms for this system. The von Neumann entropy S of a system is defined as [32]:

$$S(t) = -\mathrm{Tr}(\hat{\rho}(t)\ln\hat{\rho}(t)), \tag{7}$$

where $\hat{\rho}(t)$ is the density operator for the given quantum system. For a system in which both the entangled atoms and field start from pure state, the entropy of the whole system is always zero, i.e., $S(t) = 0$. Therefore the entropies of two interacting subsystem are precisely equal, i.e., $S_A(t) = S_F(t)$. Information about the bipartite (i.e., atomic system and field) is involved in the wave function (4) or in the total density matrix $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$. Nevertheless, the information on the atomic system solely can be obtained from the atomic reduced density matrix $\hat{\rho}(t)$ having the form:

$$\hat{\rho_A}(t) = \mathrm{Tr}_F[\hat{\rho}(t)] = \mathrm{Tr}_F[\langle\Psi|\Psi\rangle]. \tag{8}$$

At any time ($\tau > 0$) the reduced two atomic density operator for the system is given by:

$$\hat{\rho_A}(t) = \begin{pmatrix} \langle\Psi_1|\Psi_1\rangle & \langle\Psi_1|\Psi_2\rangle & \langle\Psi_1|\Psi_3\rangle & \langle\Psi_1|\Psi_4\rangle \\ \langle\Psi_2|\Psi_1\rangle & \langle\Psi_2|\Psi_2\rangle & \langle\Psi_2|\Psi_3\rangle & \langle\Psi_2|\Psi_4\rangle \\ \langle\Psi_3|\Psi_1\rangle & \langle\Psi_3|\Psi_2\rangle & \langle\Psi_3|\Psi_3\rangle & \langle\Psi_3|\Psi_4\rangle \\ \langle\Psi_4|\Psi_1\rangle & \langle\Psi_4|\Psi_2\rangle & \langle\Psi_4|\Psi_3\rangle & \langle\Psi_4|\Psi_4\rangle \end{pmatrix}, \tag{9}$$

where the eigenvalues of the reduced two atomic density operator $\hat{\rho_A}(t)$ can be obtained as follows:

$$\begin{aligned} &\pi_A^1(t) = 0, \qquad \pi_A^2(t) = 2\left[\frac{1}{6} + r^{\frac{1}{3}}\cos(\frac{\theta}{3})\right], \\ &\pi_A^3(t) = 2\left[\frac{1}{6} + r^{\frac{1}{3}}\cos\left(\frac{\theta+2\pi}{3}\right)\right], \qquad \pi_A^4(t) = 2\left[\frac{1}{6} + r^{\frac{1}{3}}\cos\left(\frac{\theta+4\pi}{3}\right)\right], \end{aligned} \tag{10}$$

where

$$\begin{aligned} &r = \frac{\sqrt{\triangle + 2 - 9A - 27B}}{54}, \qquad \theta = \tan^{-1}\frac{\triangle}{2 - 9A - 27B}, \\ &\triangle = 4(3A-1)^3 + (-2 + 9A + 27B)^2 \end{aligned} \tag{11}$$

with

$$A = \sum_{i,j=1}^{4}\sum_{i<j}^{4}\langle\Psi_i|\Psi_i\rangle\langle\Psi_j|\Psi_j\rangle - \sum_{i,j=1}^{4}\sum_{i<j}^{4}|\langle\Psi_i|\Psi_j\rangle|^2, \tag{12}$$

$$\begin{aligned} B = &-\sum_{i=1}^{4}\langle\Psi_i|\Psi_i\rangle\sum_{j<k:j,k>i}^{4}\langle\Psi_j|\Psi_j\rangle\langle\Psi_k|\Psi_k\rangle + \sum_{i=1}^{4}\langle\Psi_i|\Psi_i\rangle\sum_{j<k:j,k\neq i}^{4}|\langle\Psi_j|\Psi_k\rangle|^2 \\ &-\sum_{i\neq j\neq k:j,k>i}^{4}\langle\Psi_i|\Psi_j\rangle\langle\Psi_j|\Psi_k\rangle\langle\Psi_k|\Psi_i\rangle. \end{aligned} \tag{13}$$

So the atomic entropy of the two atoms $S_A(t)$ is given by:

$$S_A(t) = -\sum_{j=1}^{4}\pi_A^j(t)\ln[\pi_A^j(t)]. \tag{14}$$

If $S_A(t)$ takes its minimum value 0, the field and the two atoms are in a disentangled state. If $S_A(t)$ takes its maximum value, the field and the two atoms are in the maximum entangled state.

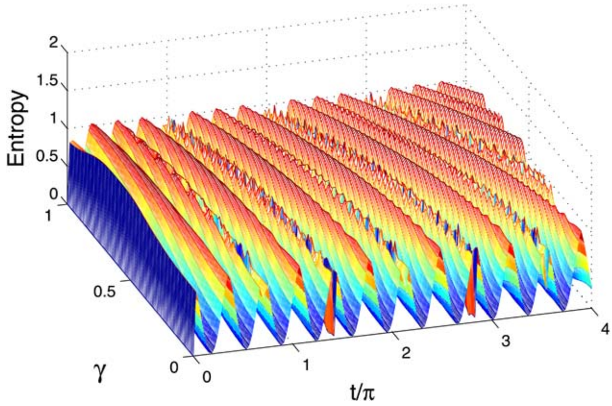

Fig. 1 The von Neumann entropy for two atoms as a function of τ and γ, when both atoms initially prepared in the exited state and the field in a coherent state for the two photons ($k = 2$) with initial mean photon number $|\alpha|^2 = 100$

3.2 Linear Entropy of the System

Here we use the linear entropy as a measure of the degree of entanglement between the field and the two atoms. The linear entropy for the atomic subsystem is defined as:

$$\xi(\tau) = 1 - \mathrm{Tr}[\rho_A^2(\tau)]. \tag{15}$$

For the system under consideration the relation (15) can be easily evaluated as:

$$\begin{aligned}\xi(\tau) = 1 &- \langle\Psi_1|\Psi_1\rangle\langle\Psi_1|\Psi_1\rangle + \langle\Psi_1|\Psi_2\rangle\langle\Psi_2|\Psi_1\rangle + \langle\Psi_1|\Psi_3\rangle\langle\Psi_3|\Psi_1\rangle \\ &+ \langle\Psi_1|\Psi_4\rangle\langle\Psi_4|\Psi_1\rangle + \langle\Psi_2|\Psi_1\rangle\langle\Psi_1|\Psi_2\rangle + \langle\Psi_2|\Psi_2\rangle\langle\Psi_2|\Psi_2\rangle \\ &+ \langle\Psi_2|\Psi_3\rangle\langle\Psi_3|\Psi_2\rangle + \langle\Psi_2|\Psi_4\rangle\langle\Psi_4|\Psi_2\rangle + \langle\Psi_3|\Psi_1\rangle\langle\Psi_1|\Psi_3\rangle \\ &+ \langle\Psi_3|\Psi_2\rangle\langle\Psi_2|\Psi_3\rangle + \langle\Psi_3|\Psi_3\rangle\langle\Psi_3|\Psi_3\rangle + \langle\Psi_3|\Psi_4\rangle\langle\Psi_4|\Psi_3\rangle \\ &+ \langle\Psi_4|\Psi_1\rangle\langle\Psi_1|\Psi_4\rangle + \langle\Psi_4|\Psi_2\rangle\langle\Psi_2|\Psi_4\rangle + \langle\Psi_4|\Psi_3\rangle\langle\Psi_3|\Psi_4\rangle \\ &+ \langle\Psi_4|\Psi_4\rangle\langle\Psi_4|\Psi_4\rangle.\end{aligned} \tag{16}$$

The case when $\xi = 0$, i.e., $\mathrm{Tr}\,\rho_A^2 = 1$ corresponds to completely disentangled atomic and field states and when the linear entropy equals to 3/4, i.e., ($\mathrm{Tr}\,\rho_A^2 = 1/4$) corresponds to maximum entanglement degree [33].

In Figs. 1 and 2, we present the time evolution of the entanglement dynamics through cooperative interaction by using two measures of entanglement, the von Neumann entropy and linear entropy to investigate the degree of entanglement when the two atoms initially prepared in the excited state and the field in a coherent state with initial mean photon numbers ($|\alpha|^2 = 100$) in the case of the two-photon ($k = 2$). These figures show that the two measures have similar behavior, and the difference between them appear in the maximum and the minimum values. So we can use any of them as a good measure of the degree of entanglement between the two atoms and the field. Furthermore, along τ-axis, the entanglement presents a behavior with periodical maximum and minimum-zero. While, this period can be changed by alternating γ.

Now we shall discuss the influence of the generation of two atoms entanglement inside a resonant microcavity under an auxiliary interaction-STE. Figs. 3, 4(a) display the time evolution of the degree of entanglement for the two measures for the two atoms without

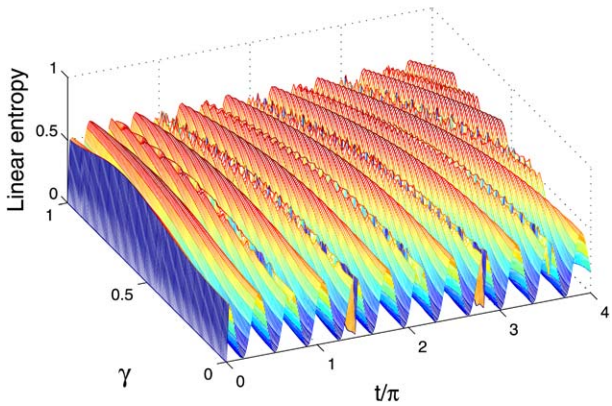

Fig. 2 The linear entropy for two atoms as a function of τ and γ, with the same initial as in Fig. 1

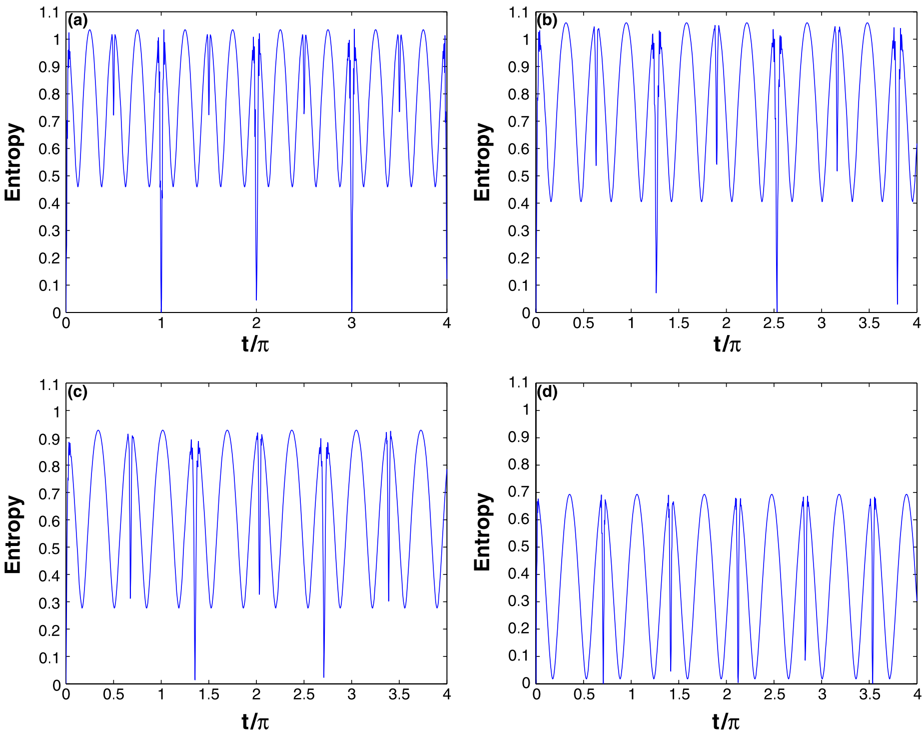

Fig. 3 The time evolution of von Neumann entropy for two atoms as a function of τ, when both atoms initially prepared in the exited state and the field in a coherent state for the two photons ($k = 2$) with initial mean photon number $|\alpha|^2 = 100$ for different values of the parameter γ: (**a**) $\gamma = 1$, (**b**) $\gamma = 0.5$, (**c**) $\gamma = 0.3$ and (**d**) $\gamma = 0$

stimulated emission (STE), i. e. $\alpha_s = 0$ ($\gamma = 1$), for the two photon ($k = 2$) with initial mean photon numbers $\bar{n} = 100$, it is observed that the two measures evolve with a period $\tau = \frac{\pi}{g}$, $S_A(t)$ evolves to zero and the field is completely disentangled from the atoms, while for $\tau = (2n-1)\frac{\pi}{4g}$, $S_A(t)$ evolves to the maximum value, and the field is strongly entangled with the two atoms. While, when STE is included, the result is apparently different. In Figs. 3, 4(b,c),

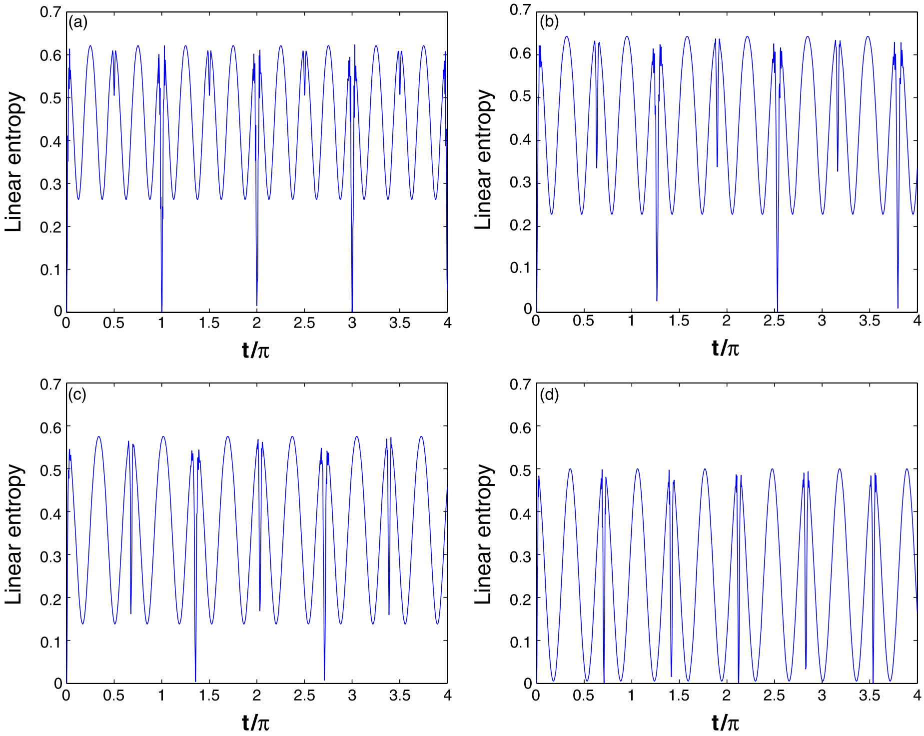

Fig. 4 The time evolution of the linear entropy for two atoms as a function of τ, with the same initial as in Fig. 3

where ($\gamma = 0.5, 0.3$) it is observed that the entanglement evolves periodically. By decreasing the parameter γ of the STE (α_s increased), the evolution period of the entanglement becomes $\frac{\pi}{g\sqrt{2(1+\gamma^2)}}$, which is larger than the case of no STE. The maximum value of the entanglement and the minimum value are decreased.

But in Figs. 3, 4(d) for $\gamma = 0$ i.e. α_s take the maximum value ($\alpha_s = 1$) the two measures have a minimum value 0 and the field is completely disentanglement from the atom. The maximum value is decreased and the evolution period of the entanglement becomes small. The entanglement decreases with respect to γ. Especially in the case when $\gamma \to 0$, where the coupling strengths g_{drv} and g_{ste} are equal, the entanglement monotonously reaches its peak. Finally we find a competition between the interaction with and without STE, while, the amplitude of entanglement without STE is larger than that with STE.

4 Conclusions

The entanglement dynamics between two atoms taking into account the stimulated emission by using two measures of entanglement has been discussed. An analytical solution for this system is obtained when both atoms are initially in the exited and the field in the coherent state. We have discussed the generation of two-atom entanglement inside a resonant micro-cavity under the stimulated emission. It has been shown that the two measures have typically information on the entanglement dynamics and the entanglement decreases due to stimulated emission. We find a competition between the interaction with and without STE, while, the amplitude of entanglement without STE is larger than that with STE.

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